A note on 'Nonexistence of self-similar singularities for the 3D incompressible Euler equations'

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Abstract

In this brief note we show that the author's previous result in [1] on the nonexistence of self-similar singularities for the 3D incompressible Euler equations implies actually the nonexistence of 'locally self-similar' singular solution, which has been sought by many physicists.

Nonexistence of locally self-similar solution

We are concerned here on the following Euler equations for the homogeneous incompressible fluid flows on $\mathbb{R}^3$.

$$(E)\begin{cases} \dfrac{\partial v}{\partial t} + (v\cdot\nabla)v = -\nabla p, \quad (x,t)\in\mathbb{R}^3\times(0,\infty) \\ \quad \mathrm{div}\ \ v = 0, \quad (x,t)\in\mathbb{R}^3\times(0,\infty) \\ v(x,0) = v_0(x), \quad x\in\mathbb{R}^3 \end{cases}$$

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where $v = (v_1, v_2, v_3)$, $v_j = v_j(x,t)$, $j = 1,2,3$, is the velocity of the flow, $p = p(x,t)$ is the scalar pressure, and v_0 is the given initial velocity, satisfying $\mathrm{div}\ v_0 = 0$.

The system (E) has the scaling property that if (v,p) is a solution of the system (E), then for any $\lambda > 0$ and $\alpha \in \mathbb{R}$ the functions

$$v^{\lambda,\alpha}(x,t) = \lambda^\alpha v(\lambda x, \lambda^{\alpha+1} t), \quad p^{\lambda,\alpha}(x,t) = \lambda^{2\alpha} p(\lambda x, \lambda^{\alpha+1} t) \tag{1}$$

are also solutions of (E) with the initial data $v_0^{\lambda,\alpha}(x) = \lambda^\alpha v_0(\lambda x)$.

This scaling property leads to the following definition of self-similar blowing up solution of (E):

Definition 1 *A solution $v(x,t)$ of the solution to (E) is called a self-similar blowing up solution if there exist $\alpha > -1$, $T_* > 0$ and a solenoidal vector field V defined on $\mathbb{R}^3$ such that*

$$v(x,t) = \frac{1}{(T_* - t)^{\frac{\alpha}{\alpha+1}}} V\left(\frac{x}{(T_* - t)^{\frac{1}{\alpha+1}}}\right) \quad \forall (x,t) \in \mathbb{R}^3 \times (-\infty, T_*). \tag{2}$$

The above definition is apparently 'global' in the sense that the self-similar representation of solution in (2) should hold for all space-time points in $\mathbb{R}^3 \times (-\infty, T_*)$. On the other hand, many physicists have been trying to seek a 'locally self-similar' solution of the 3D Euler equations(see e.g. [4] and the references therein). We formulate the precise definition of this in the following.

Definition 2 *A solution $v(x,t)$ of the solution to (E) is called a locally self-similar blowing up solution near a space-time point $(x_*, T_*) \in \mathbb{R}^3 \times (-\infty, +\infty)$ if there exist $r > 0$, $\alpha > -1$ and a solenoidal vector field V defined on $\mathbb{R}^3$ such that the representation*

$$v(x,t) = \frac{1}{(T_* - t)^{\frac{\alpha}{\alpha+1}}} V\left(\frac{x - x_*}{(T_* - t)^{\frac{1}{\alpha+1}}}\right) \ \forall (x,t) \in B(x_*, r) \times (T_* - r^{\alpha+1}, T_*) \tag{3}$$

holds true, where $B(x_*, r) = \{x \in \mathbb{R}^3 \,|\, |x - x_*| < r\}$.

We have the following relation between the two notions of self-similar blowing up solutions.

Theorem 1 *The nonexistence of self-similar solution of the 3D Euler equations in the sense of Definition 1 implies the nonexistence of locally self-similar solution in the sense of Definition 2.*

Combining Theorem 1 with the main theorem in [1](Theorem 1.1), we have the following corollary.

Corollary 1 *Suppose there exists a locally self-similar blowing up solution of the 3D Euler equations in the form (3). If there exists $p_1 > 0$ such that $\Omega = \mathrm{curl}\, V \in L^p(\mathbb{R}^3)$ for all $p \in (0, p_1)$, then necessarily $\Omega = 0$. In other words, there exists no nontrivial locally self-similar solution to the 3D Euler equation if the vorticity Ω satisfies the integrability condition above.*

Proof of Theorem 1. We assume there exists a locally self-similar blowing up solution $v(x,t)$ in the sense of Definition 2. The proof of the theorem follows if we prove the existence of self-similar blowing up solution in the sense of Definition 1 based on that assumption. By translation in space-time variables, we can rewrite the velocity in (3) as

$$v(x,t) = \frac{1}{t^{\frac{\alpha}{\alpha+1}}} V\left(\frac{x}{t^{\frac{1}{\alpha+1}}}\right) \quad \text{for} \quad (x,-t) \in B(0,r) \times (-r^{\alpha+1}, 0). \tag{4}$$

We observe that, under the scaling transform (1), we have the invariance of the representation,

$$v(x,t) \mapsto v^{\lambda,\alpha}(x,t) = \lambda^\alpha v(\lambda x, \lambda^{\alpha+1} t) = \frac{1}{t^{\frac{\alpha}{\alpha+1}}} V\left(\frac{x}{t^{\frac{1}{\alpha+1}}}\right) (= v(x,t)),$$

while the region of space-time, where the self-similar form of solution is valid, transforms according to

$$B(0,r) \times (-r^{\alpha+1}, 0) \mapsto B(0, r/\lambda) \times \left(-(r/\lambda)^{\alpha+1}, 0\right).$$

We set $\lambda = 1/n$, and define the sequence of locally self-similar solutions $\{v^n(x,t)\}$ by $v^n(x,t) := v^{\frac{1}{n},\alpha}(x,t)$ with $v^1(x,t) = v(x,t)$. In the above we find that

$$v^n(x,t) = \frac{1}{t^{\frac{\alpha}{\alpha+1}}} V\left(\frac{x}{t^{\frac{1}{\alpha+1}}}\right) \quad \text{for} \quad (x,-t) \in B(0,nr) \times (-(nr)^{\alpha+1}, 0),$$

and each $v^n(x,t)$ is a solution of the Euler equations for all $(x,t) \in \mathbb{R}^3 \times (-\infty, 0)$. Let us define $v^\infty(x,t)$ by

$$v^\infty(x,t) = \frac{1}{t^{\frac{\alpha}{\alpha+1}}} V\left(\frac{x}{t^{\frac{1}{\alpha+1}}}\right) \quad \text{for} \quad (x,-t) \in \mathbb{R}^3 \times (-\infty, 0).$$

Given a compact set $K \subset \mathbb{R}^3 \times (-\infty, 0)$, we observe that $v^n \to v^\infty$ as $n \to \infty$ on K in any strong sense of convergence. Indeed, for sufficiently large $N = N(K)$, $v^n(x,t) \equiv v^\infty(x,t)$ for $(x,t) \in K$, if $n > N$. Hence, we find that $v^\infty(x,t)$ is a solution of the Euler equations for all $(x,t) \in \mathbb{R}^3 \times (-\infty, 0)$, which is a self-similar blowing up solution, after translation in time, in the sense of Definition 1. $\square$

We note that the above proof obviously works also for the self-similar solutions of the other equations considered in [1] and Leray's self-similar solutions of the Navier-Stokes equations(see [2] for the problem, and [3] for the rule-out of the 'global' self-similar solution).

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References

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